MATH 2020A Tutorial 1

I. Review techniques in Single-variable integration.

1. Integral by parts formula for indefinite integrals,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx.$$

Ex. 1 Find $\int \ln x\,dx$.

Solution.
$$\int \ln x\,dx = \int \ln x \cdot 1\,dx = x \cdot \ln x - \int \frac{1}{x} \cdot x\,dx$$
$$= x\ln x - x + C. \quad (\text{For some constant } C).$$

- For definite integrals,

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Ex. 2 Find $\int_0^4 xe^{-x}dx$.

Solution.
$$\int_0^4 xe^{-x}dx = -xe^{-x}\Big|_0^4 - \int_0^4(-e^{-x})dx$$
$$= -4e^{-4} + \int_0^4 e^{-x}\,dx$$
$$= -4e^{-4} - e^{-4} + 1 = 1 - 5e^{-4}.$$

2. Integral by substitution

most common trigonometric substitution are $x = a\tan\theta$, $x = a\sin\theta$ and $x = a\sec\theta$ for $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$.

Ex 1. Find $\int \frac{dx}{\sqrt{4+x^2}}$

Solution. Set $x = 2\tan\theta$, then $dx = 2\sec^2\theta\,d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Then
$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\,d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\,d\theta}{|\sec\theta|} = \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

Ex2. Evaluate $\int_0^2 x e^{x^2} dx$.

Solution. Let $u = x^2$, then $du = 2x dx$.

Thus $\int_0^2 x e^{x^2} dx = \int_0^2 (\frac{1}{2} e^{x^2})(2x dx) = \int_0^4 \frac{1}{2} e^u du = \frac{1}{2} e^u \Big|_0^4$

$= \frac{1}{2}(e^4 - 1)$

3. Improper integrals

① Improper integrals of Type I (Infinite in domain):

If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

Ex. Evaluate $\int_1^\infty \frac{\ln x}{x^2} dx$. Determine if it is finite.

Solution. $\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{b \to \infty} \int_1^b \frac{\ln x}{x^2} dx = \lim_{b \to \infty} \left\{ \ln x \cdot (-\frac{1}{x}) \Big|_1^b - \int_1^b (-\frac{1}{x})(\frac{1}{x}) dx \right\}$

$= \lim_{b \to \infty} (-\frac{\ln b}{b} - \frac{1}{x} \Big|_1^b) = \lim_{b \to \infty} (-\frac{\ln b}{b} - \frac{1}{b} + 1)$

$= -\lim_{b \to \infty} \frac{\ln b}{b} - 0 + 1$

$= -\lim_{b \to \infty} \frac{\frac{1}{b}}{1} - 0 + 1 = 1 < \infty$.

② Improper integrals of Type II. (Infinite in range):

If $f(x)$ is continuous on $(a, b]$ and discts at a, then

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

If $f(x)$ is cts on $[a, b)$ and discts at b, then

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx.$$

If $f(x)$ is discts at c, $c \in (a,b)$, and cts elsewhere, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Ex. Evaluate $\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}}$.

Solution: This integral is discts at $x=1$.

Thus $$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} + \int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}}.$$

$$\int_0^1 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{b\to 1^-} 3(x-1)^{\frac{1}{3}}\Big|_0^b = \lim_{b\to 1^-} [3(b-1)^{\frac{1}{3}}+3] = 3$$

$$\int_1^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{a\to 1^+} \int_a^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = \lim_{a\to 1^+} 3(x-1)^{\frac{1}{3}}\Big|_a^3$$

$$= \lim_{a\to 1^+} [3(3-1)^{\frac{1}{3}} - 3(c-1)^{\frac{1}{3}}] = 3\sqrt[3]{2}$$

So $$\int_0^3 \frac{dx}{(x-1)^{\frac{2}{3}}} = 3 + 3\sqrt[3]{2}.$$

③ Improper double integrals

Like single integrals, $\iint_D f\,dA$ is an improper integral if D is an unbounded region or f is an unbounded function.

Ex. Evaluate $\int_0^\infty \int_0^\infty xy e^{-x^2-y^2}\,dy\,dx$

Solution. $$\int_0^\infty \int_0^\infty xy e^{-x^2-y^2}\,dy\,dx = \lim_{(a,b)\to(\infty,\infty)} \int_0^a \int_0^b xy e^{-x^2-y^2}\,dy\,dx$$

$$= \lim_{(a,b)\to(\infty,\infty)} \frac{1}{4}(1-e^{-a^2})(1-e^{-b^2})$$

$$= \frac{1}{4} < \infty.$$

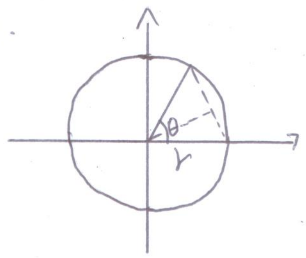
II Average Value

1. Average value for single variable function over $[a, b]$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Ex. Find the average length of chords of a circle with radius r

Solution.



The length of the chord

$$= 2r\sin\frac{\theta}{2}$$

$$\text{Average value} = \frac{1}{2\pi-0}\int_0^{2\pi} 2r\sin\frac{\theta}{2}\,d\theta$$

$$= \frac{2r}{2\pi}\int_0^{2\pi}\sin\frac{\theta}{2}\,d\theta$$

$$= \frac{2r}{\pi}\int_0^{2\pi}\sin\frac{\theta}{2}\,d\frac{\theta}{2}$$

$$= \frac{4r}{\pi}$$

2. Average value for double variables function:

$$\frac{1}{\text{area of }\Omega}\iint_\Omega f\,dA.$$

Ex. Find the average value of the paraboloid $z = x^2+y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

Solution:

$$\frac{1}{2\times 2}\int_0^2\int_0^2 x^2+y^2\,dx\,dy$$

$$= \frac{1}{4}\int_0^2 \frac{x^3}{3}+xy^2\Big|_0^2\,dy$$

$$= \frac{1}{4}\int_0^2 \frac{8}{3}+2y^2\,dy$$

$$= \frac{1}{4}\left(\frac{8}{3}y+\frac{2}{3}y^3\right)\Big|_0^2$$

$$= \frac{8}{3}.$$

III. Fubini's Theorem

Over bounded, rectangle regions:

If $f(x,y)$ is cts throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

Over bounded, nonrectangular regions:

Let $f(x,y)$ be cts on a region R.

If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 cts. on $[a,b]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 cts. on $[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Ex. Evaluate $\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx$

Solution.
$$\int_0^\pi \int_x^\pi \frac{\sin y}{y}\,dy\,dx = \int_0^\pi \int_0^y \frac{\sin y}{y}\,\cancel{dy\,dx}\ dx\,dy$$
$$= \int_0^\pi \sin y\,dy = 2.$$